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GENERATING THE MAPPING CLASS GROUP (AN ALGEBRAIC APPROACH)

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Abstract

We give an algebraic proof of the fact that a generating set of the mapping class group $M_{g,1}$ ($g \geq 3$) may be obtained by replicating a generating set of $M_{2,1}$.

1. Introduction. We denote by $F(S)$ the free group generated by the subset S of the set of symbols $X = a_1, b_1, a_2, b_2, \dots$, and put $F_k = F(X_k)$, where X_k consists of the first k elements of X (all groups $F(S)$ are considered as subgroups of $F(X)$). $L(X)$, the set of letters, is defined to be $X \cup X^{-1}$, i.e. the set $a_1, \overline{a}_1, b_1, \overline{b}_1, \dots$, where $\overline{a}_1$ denotes a_1^{-1}, etc.; for $S \subset X$, the set of letters $L(S)$ of $F(S)$ is $F(S) \cap L(X)$. For $w \in F(X)$, $L(W)$ is the set of letters occurring in the reduced form of w.

We put $\mathcal{A}(S) = \operatorname{Aut} F(S)$, with $\mathcal{A}_k$ for $\mathcal{A}(X_k)$, and denote by Π_g the element of F_{2g} given by $\Pi_g = \prod_{i=1}^{g} [a_i, b_i]$, where $[a_i, b_i] = a_i b_i \overline{a}_i \overline{b}_i$. The group $M(\Pi_g)$ is defined by

$$M(\Pi_g) = \{\theta \in \mathcal{A}_{2g};\ \Pi_g \theta = \Pi_g\}.$$

Let ρ_g denote conjugation in F_{2g} by the element Π_g. The subgroup N_g of $M(\Pi_g)$ generated by ρ_g is central, and the quotient $M_{g,1} = M(\Pi_g)/N_g$ may be described as an (orientation preserving) *(algebraic) mapping class group*. It was shown in [**9**] that $M_{g,1}$ is finitely presented, though computation of an explicit presentation valid for all g was beyond the scope of the results of [**9**]. Such a presentation of the geometric mapping class group was found by Wajnryb [**11**]. Since the geometric and algebraic mapping class groups are known to coincide (see, e.g., the remarks and

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references in [**3**]), this provides a presentation for our $M_{g,1}$. Wajnryb's work is geometrically based, as is earlier work on generating sets by Dehn [**2**], Lickorish [**6**] and Humphries [**5**].

The present paper has the modest object of providing a purely algebraic method for obtaining a generating set of $M(\Pi_g)$, and hence of $M_{g,1}$. Thus we define the groups $M(r,g)$, for $1 \le r \le g-1$, by

$$M(r,g) = \{\theta \in \mathcal{A}\{a_r, b_r, a_{r+1}, b_{r+1}\};\ [a_r,b_r][a_{r+1},b_{r+1}]\theta = [a_r,b_r][a_{r+1},b_{r+1}]\}.$$

Clearly each $M(r,g)$ is an isomorphic copy of $M(\Pi_2)$, and each $M(r,g)$ is naturally embedded in $M(\Pi_g)$, as is each $M(\Pi_r)$, for $r < g$. We will show

Theorem. *Let G_r be a generating set of $M(r,g)$, $1 \le r \le g-1$. Then $\bigcup_{r=1}^{g-1} G_r$ is a generating set of $M(\Pi_g)$.*

It only remains, in order to fulfil our objective, to find a generating set G_2 of $M(\Pi_2)$. We discuss this after the proof of the theorem.

We assume below that the reader is familiar with the notation and results of [**8**] and [**9**] (see also [**7**]). In addition, we will need the following definition.

Let $S \subset L(X_g)$, and $\theta \in \mathcal{A}_g$. We say that θ *involves only the letters* of S if, writing S_1 for $S^{\pm 1} \cap X_g$, there exists $\varphi \in \mathcal{A}(S_1)$ such that θ and φ agree on S_1 and θ is the identity on $X_g - S_1$.

2. Preliminary results. The following result was proved by Shenitzer in [**10**].

Lemma 1. *Let W be a minimal element of F_k with $|W| > 1$. Let $(A;a)$ be a T_2 in $\mathcal{A}_k$, with $a, \overline{a}$ not in $L(W)$ and with $A \cap L(W)$ not the empty set. Then $|W(A;a)| \ge |W| + 2$.*

As a consequence of this we have

Corollary 2. (1) *A product $W_1 W_2 \cdots W_r$ of disjoint minimal elements of F_k is minimal if, and only if, $|W_i| \ge 2$, $1 \le i \le r$.*

(2) *Two equivalent minimal words involve the same number of generators.*

(3) *If W is minimal, $|W| > 1$ and $(A;a) = (x_1, \ldots, x_j, a; a)$ is a T_2 such that $A \cap L(W)$ is non-empty and $|W(A;a)| \le |W|$, then W must contain a subword $x_i \overline{a}$ or $a \overline{x}_i$ for some i, $1 \le i \le j$.*

Proof: Parts (1) and (2) were proved by Shenitzer in [**10**]. An immediate consequence of these is the fact that for any $S \subset X$ and $W \in F(S)$, W is minimal in $F(S)$ if, and only if, W is minimal in $F(X)$.

Now suppose W, $(A; a)$ satisfy the conditions of (3), and no subword of the desired form exists. Let W' be the unreduced word obtained from W by replacing each letter b in W by $b(A; a)$. It is known [**4**] that $w(A; a)$ is obtained from W' by deleting all subwords of W' of the form $a\overline{a}$. Since W contains no subword of the form $x_i\overline{a}$ or $a\overline{x}_i$, the a and $\overline{a}$ symbols in any subword $a\overline{a}$ of W' must both be 'new'. Now let W_1 be obtained from W by replacing each $a, \overline{a}$ by $x, \overline{x}$ respectively, where x is a letter not in $L(W) \cup A \cup A^{-1}$. From the above remark, it is clear that $|W_1(A; a)| = |W(A; a)| \leq |W|$. However, this contradicts Lemma 1, and so proves (3). ■

It follows from (1) that Π_g is minimal, since $[a_i, b_i]$ is clearly minimal. We denote by $m(\Pi_g)$ the set of minimal equivalents of Π_g in F_{2g}. If $V \in m(\Pi_g)$ then we observe that V must contain exactly one occurrence of each letter in $L(X_{2g})$. Now if $V \in m(\Pi_g)$ has a subword $x\overline{y}$, where $x, y \in L(X_{2g})$, then it is clear, since V contains one occurrence of each of $x, \overline{x}$, that $V(x, y; y)$ belongs to $m(\Pi_g)$ (as does $V(y, x; x)$). Combining this observation with (3) of Corollary 2, we obtain

Corollary 3. *Let $V \in m(\Pi_g)$ and let $(A; a) = (y_1, \ldots, y_r, a; a) \in \mathcal{A}_{2g}$ be such that $V(A; a) \in m(\Pi_g)$. Then there is a permutation $\sigma \in S_r$ such that*

$$V(y_{\sigma(1)}, a; a) \cdots (y_{\sigma(i)}, a; a) \in m(\Pi_g)$$

for $1 \leq i \leq r$.

We next prove

Lemma 4. *Let r, k be positive integers with $r < k$ and let $Y = X_k - X_r$. Let U, V, W be such that $U, W \in F_r$, $L(V) \cup L(V^{-1}) = Y \cup Y^{-1}$ and V is minimal. If $\beta \in \mathcal{A}_k$ is such that $x_i\beta = x_i$, $1 \leq i \leq r$, where $X_r = \{x_1, \ldots, x_r\}$, and $(WV)\beta = UV$, then $U = W$ and β involves only the letters of Y.*

Proof: Let $W_1 = U^{-1}W$, so that $(W_1V)\beta = V$. We put

$$Z = \{x_1, \ldots, x_r, \ldots, x_1, \ldots, x_r, W_1V\},$$

where Z contains N occurrences of the r-tuple $Z_1 = (x_1, \ldots, x_r)$, and N is chosen so $N > |W_1V|$. Then Z is mapped by β to $Z_2 = \{Z_1, \ldots, Z_1, V\}$.

Since $|Z| \geq |Z_2|$, there exists (see [**8**], [**9**]) a factorisation $\beta = P_1' \cdots P_s'$, where $P_1', \dots, P_s' \in \mathcal{W}$, and an integer t, $1 \leq t \leq s$, such that

$$|ZP_1' \cdots P_i'| < |ZP_1' \cdots P_{i-1}'|, \quad i \leq t, \tag{1}$$

and

$$|ZP_1' \cdots P_i'| = |Z_2|, \quad i \geq t. \tag{2}$$

Each P_i' with $i \leq t$ must be a T_2. Now for any tuple Z_3, type one T and type two P, if $|Z_3| = |Z_3(TP)|$, then

$$|Z_3| = |Z_3(TPT^{-1})| = |Z_3(TPT^{-1})T|,$$

and $TPT^{-1} \in T_2$. Using this observation, we can modify the original factorisation of β to obtain $\beta = P_1 \cdots P_l T$ where $P_1, \dots, P_l$ are T_2's, T is a T_1 (possibly the identity) and (1), (2) hold with $P_1, \dots, P_l$ in place of $P_1', \dots, P_s'$. From the choice of Z it is easy to see that no P_i can increase the length of any one of $x_1, \dots, x_r$, and hence each P_i and T must fix all of $x_1, \dots, x_r$.

If P_1 has multiplier from $L(Y)$, then

$$|(W_1V)P_1| = |W_1(VP_1)| \geq |W_1V|,$$

since V is minimal and no cancellation occurs between W_1 and VP_1. In view of (1) it follows that $W_1 = 1$ in this case.

If P_1 has multiplier from $L(X_r)$, then by Lemma 1 $|VP_1| \geq |V| + 2$; moreover, in $(W_1V)P_1 = W_1(VP_1)$, at most one cancellation can occur between W_1 and VP_1, so that $|(W_1V)P_1| \geq |W_1V|$ and again we must have $W_1 = 1$.

Hence we have shown that $W_1 = 1$. It now follows from Lemma 1, as above, that P_1 cannot have multiplier from $L(X_r)$, and the same argument shows, inductively, that no P_i can have multiplier from $L(X_r)$. Since each P_i fixes X_r pointwise, so must T. This proves the lemma. ■

Definition. Let $V \in m(\Pi_g)$, $A \subset L(X_{2g})$, $A \cap A^{-1} = \varnothing$, $|A| = 2r$ for some integer $r \geq 1$. We say that A is *interlocked* in V if the "quotient word" $V(A)$ obtained by deleting all letters in $L(X_{2g}) - (A + A^{-1})$ from V is a minimal equivalent of Π_r.

Let $V \in m(\Pi_g)$ have reduced form $V = QxR\overline{x}S$, for some letter x. Then there is $y \in R$ (i.e. letter y which is a subword of R) such that $\overline{y} \notin R$, for otherwise conjugation of the generators occurring in R by x would reduce the length of V. Hence for each $x \in V$ there is a $y \in V$ such that x, y are interlocked in V.

We now observe

Lemma 5. *Let A be interlocked in V and let $\theta \in \mathcal{A}_{2g}$ be such that $V\theta \in m(\Pi_g)$ and $G\theta = G$, where G is the normal closure in F_{2g} of $L(X_{2g}) - (A + A^{-1})$. Then A is interlocked in $V\theta$.*

Proof: For ease of notation we suppose that $A = X_{2r}$. Let p be the projection $p : F_{2g} \to F_{2g}/G = F_{2r}$. Since $G\theta = G$, θ induces an automorphism θ_1 of F_{2r} and $p\theta_1 = \theta p$, so that

$$Vp\theta_1 = V(A)\theta_1 = V\theta p = (V\theta)(A).$$

Now $V(A) \in m(\Pi_r)$ since A is interlocked in V. Thus $(V\theta)(A)$ is an automorphic image of Π_r and so belongs to $m(\Pi_r)$, since it has length $4r$. Hence A is interlocked in $V\theta$. ■

3. The complex K_g. Let K_g be the complex for Π_g constructed in [**9**]; i.e. $K_g^0 = m(\Pi_g)$, K_g^1 is K_g^0 with a directed edge labelled $(V_1, V_2; P)$ joining vertex V_1 to V_2 whenever $P \in \mathcal{W}$ is such that $V_1 P = V_2$, and K_g is K_g^1 with a finite set of 2-cells attached. It was shown in [**9**] that there is an isomorphism $\kappa : \pi_1(K_g, \Pi_g) \to M(\Pi_g)$, and that the isomorphism is the natural one, i.e. is induced by the homomorphism κ from the groupoid of paths in K_g to $\mathcal{A}_{2g}$ whose effect on a path p in K_g,

$$p = (V_1, V_2; P_1), (V_2, V_3; P_2), \ldots, (V_{s-1}, V_s; P_{s-1}),$$

is given by $p\kappa = P_1 P_2 \cdots P_{s-1}$.

Let $V \in m(\Pi_g)$ be such that x, y are interlocked in V. Then there is a (unique) $T \in T_1$ with T involving only x and y such that $VT = V_1 = AxByC\overline{x}D\overline{y}E$ (where the expression given for V_1 is reduced). Now let E have reduced form $x_1 x_2 \cdots x_t$. Then

$$V(\overline{y}, \overline{x}_1; \overline{x}_1)(\overline{y}, \overline{x}_2; \overline{x}_2) \cdots (\overline{y}, \overline{x}_i; \overline{x}_i) \in m(\Pi_g), \quad 0 \le i \le t.$$

The product $\mu_1 = (\overline{y}, \overline{x}_1; \overline{x}_1) \cdots (\overline{y}, \overline{x}_t; \overline{x}_t)$ maps V_1 to $V_2 = AxBEyC\overline{x}D\overline{y}$, and may be denoted by $\mu_1 : y \to Ey$, since μ_1 fixes each letter other than $y, \overline{y}$. The factorisation given for μ_1 yields a path p_1 in K_g of length r from V_1 to V_2, with $p_1\kappa = \mu_1$. Now define μ_2, μ_3 and μ_4 by $\mu_2 : x \to x\overline{BE}$, $\mu_3 : y \to y\overline{CBE}$, $\mu_4 : x \to DCBEx$. Then $V_2\mu_2 = V_3 = AxyCBE\overline{x}D\overline{y}$, $V_3\mu_3 = V_4 = Axy\overline{x}DCBE\overline{y}$, $V_4\mu_4 = V_5 = ADCBExy\overline{x}\,\overline{y}$. Each μ_i has a factorisation similar to that given for μ_1, and a corresponding path p_i in K_g with $p_i\kappa = \mu_i$. We put $\mu = T\mu_1\mu_2\mu_3\mu_4$ and let p be the path $(V_1, V_2; T)$, p_1, p_2, p_3, p_4, so that $p\kappa = \mu$. The μ_i are instances of the familiar 'cut and paste' operations, and we shall refer to both p and μ as *the CP* operation on x, y taking V to $ADCBExy\overline{x}\,\overline{y}$. We note that μ

moves only x and y. Now $ADCBE$ is minimal, involves exactly $2g-2$ elements of X, and each of these occur once with exponent one and once with exponent minus one. It follows easily from this that there is a sequence of CP operations which involve only the generators occurring in $ADCBE$ and which map $ADCBE$ to $\Pi_{g-1}J$, where $J \in T_1$.

We now observe

Lemma 6. *Let a, b be interlocked in $V \in m(\Pi_g)$, $g \geq 2$. Let $x \in L(X_{2g})$ be such that $x \notin \{a, \overline{a}, b, \overline{b}\}$. Then there is $y \in V$ such that $\{a, b, x, y\}$ is interlocked in V.*

Proof: Let μ be the CP's on a, b taking V to $V_1 = U[a, b]$. Then $x \in U$ and there is $y \in U$ such that x, y are interlocked in U. Clearly $\{a, b, x, y\}$ is interlocked in V_1, and so by Lemma 5, is interlocked in V. ■

We now specify for each $V \in m(\Pi_g)$ a path τ_V from V to Π_g. For $g = 1$ and $V \in m(\Pi_1)$, there exists a unique type one $T_V \in \mathcal{A}_2$ such that $VT_V = \Pi_1$; we define τ_V to be (V_1, Π_1, T_V). Now suppose that $g > 1$ and that τ_V has been defined for all $V \in m(\Pi_r)$, $1 \leq r < g$. Let $V \in m(\Pi_g)$ and write $V = A'xB'yC'\overline{x}D'\overline{y}$, where x is the first letter to the left of y in V such that x and y are interlocked in V. Let $\theta_V = \theta_1\theta_2$, where θ_1 is the type one interchanging b_g and y, and θ_2 is the type one interchanging $x\theta_1$ and a_g. Then $V\theta_V = Aa_gBb_gC\overline{a}_gD\overline{b}_g$. We call θ_V the correcting permutation on V. Now let μ_V be the CP's on a_g, b_g taking $V\theta_V$ to $ADCBa_gb_g\overline{a}_g\overline{b}_g$. From above, we know that $ADCB \in m(\Pi_{g-1})$ and so a path, call it γ_V, has already been defined from $ADCB$ to Π_{g-1} in K_{g-1}. Taking the obvious interpretation of γ_V as a path in K_g, we define τ_V to be the path $(V, V\theta_V; \theta_V), \mu_V, \gamma_V$. We shall denote the images of the paths τ_V, μ_V, γ_V under κ by the same symbols in what follows.

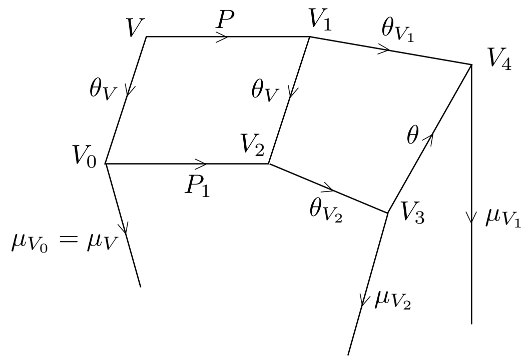
Now it is clear that $\pi_1(K_g, \Pi_g)$ is generated by the classes of the set of paths $\tau_V^{-1}, e, \tau_{V_1}$, where V ranges over the points of K_g and $e = (V, V_1; P)$ ranges over the edges beginning at V. Moreover, it follows easily from Corollary 3 that we can restrict e to range over the edges $(V, V_1; P)$ where P is a Nielsen automorphism, in fact either $P \in T_1$ or P is of the form $(a, b; b)$, where $a\overline{b}$ or $b\overline{a}$ is a subword of V.

It follows that $M(\Pi_g)$ is generated by all $\tau_V^{-1}P\tau_{V_1}$, i.e. by all $\gamma_V^{-1}\mu_V^{-1}\theta_V^{-1}P\theta_{V_1}\mu_{V_1}\gamma_{V_1}$, where here V ranges over $m(\Pi_g)$, P ranges over the Nielsen automorphisms described above, $V_1 = VP$ and $\gamma_V, \gamma_{V_1}, \mu_V, \mu_{V_1}$, θ_V, θ_{V_1} are as defined above.

We observe that if $P \in T_1$ then $\theta_V P\theta_{V_1} \in T_1$ and does not involve a_g or b_g. Also, if $P = (a, b; b)$, then

$$\theta_V^{-1}P\theta_{V_1} = (a\theta_V, b\theta_V; b\theta_V)\theta_V^{-1}\theta_{V_1} = P_1\theta_{V_2}\theta,$$

where $P_1 = (a\theta_V, b\theta_V; b\theta_V)$, $V_2 = V\theta_V P_1$ and $\theta = \theta_{V_2}^{-1}\theta_V^{-1}\theta_{V_1}$. The portion of K_g relating to this will look like



where $V_0 = V\theta_V$, $V_3 = V_2\theta_{V_2}$, $V_4 = V_1\theta_{V_1}$. We see that

$$\begin{aligned}\tau_V^{-1}P\tau_{V_1} &= \tau_V^{-1}\mu_V^{-1}\theta_V^{-1}P\theta_{V_1}\mu_{V_1}\gamma_{V_1}\\ &= (\gamma_V^{-1}\mu_V^{-1}P_1\theta_{V_2}\mu_{V_2}\gamma_{V_2})(\gamma_{V_2}^{-1}\mu_{V_2}^{-1}\theta\mu_{V_1}\gamma_{V_1})\\ &= (\tau_{V_0}^{-1}P_1\tau_{V_2})(\tau_{V_3}^{-1}\theta\tau_{V_4}).\end{aligned}$$

We note that $\theta \in T_1$ and does not involve a_g or b_g.

From the above observations we see that $M(\Pi_g)$ is generated by the set of all $k(V,N) = \gamma_V^{-1}\mu_V^{-1}N\mu_{V_1}\gamma_{V_1}$, where V ranges over the elements of $m(\Pi_g)$ with $\theta_V = 1$, N is either a type one not involving a_g or b_g (in which case $\theta_{VN} = 1$) or $N = P\theta_{VP}$ where P is a type two Nielsen automorphism, and $VN = V_1$.

We say that a $k(V,N)$ is *nice* if there is a set $S = \{a_g, b_g, x, y\}$ of letters such that S is interlocked in V and N involves only the elements of S. We note that if $k(V,N)$ is nice then, by Lemma 5, the corresponding set S is interlocked in VN. ■

The following is the key result in proving the theorem.

Lemma 7. *Let $k(V,N)$ be nice. Then $k(V,N) = k_1hk_2$, where $h \in M(g-1,g)$ and $k_1, k_2 \in M(\Pi_{g-1})$.*

Proof: We may assume that $g \geq 3$. Let S be a set such that $S = \{a_g, b_g, x, y\}$ and S is interlocked in V. Let $V = Aa_gBb_gC\overline{a}_gC\overline{b}_g$, $V_1 = A_1a_gB_1b_gC_1\overline{a}_gD_1\overline{b}_g$. Then, by Lemma 5, x, y are interlocked in both $ADCB$ and $A_1D_1C_1B_1$. Let η be the CP's on x, y taking $ADCB$ to (say)

$U_0'[x,y]$, and η_1 the CP's on x, y taking $A_1D_1C_1B_1$ to (say) $U_1'[x,y]$. Then $h' = \eta^{-1}\mu_V^{-1}N\mu_{V_1}\eta_1$ maps $U_0'[x,y][a_g,b_g]$ to $U_1'[x,y][a_g,b_g]$, and fixes each element of $L(X_{2g}) - S$. Hence, by Lemma 4, h' involves only x, y, a_g and b_g, and $U_0' = U_1'$. Let τ be a type one not involving a_g or b_g, such that $a_{g-1}\tau = x$ and $b_{g-1}\tau = y$. Let $U_0'\tau^{-1} = U_0$, so that $\{U_0'[x,y]\}\tau^{-1} = U_0[a_{g-1},b_{g-1}]$. Clearly $U_0 \in M(\Pi_{g-2})$. Choose $\lambda \in \mathcal{A}_{2g-4}$ such that $U_0\lambda^{-1} = \Pi_{g-2}$.

Now

$$\begin{aligned} k(V,N) &= \gamma_V^{-1}\mu_V^{-1}N\mu_{V_1}\gamma_{V_1} \\ &= (\gamma_V^{-1}\eta\tau^{-1}\lambda^{-1})(\lambda\tau h'\tau^{-1}\lambda^{-1})(\lambda\tau\eta_1^{-1}\gamma_{V_1}) \\ &= k_1hk_2 \end{aligned}$$

say. From their definition, it is clear that $k_1, k_2 \in M(\Pi_{g-1})$. Since h' involves only x, y, a_g, b_g, it follows that $\tau h'\tau^{-1}$ involves only a_{g-1}, b_{g-1}, a_g and b_g, and so commutes with λ. Hence $h = \tau h'\tau^{-1} \in M(g-1,g)$. ■

4. Proof of the Theorem. The theorem follows immediately from

Lemma 8. *For each $k(V,N)$ there exist $k_1, k_2 \in M(\Pi_{g-1})$ and $h \in M(g-1,g)$ such that $k(V,N) = k_1hk_2$.*

Proof: Let $V = Aa_gBb_gC\overline{a}_gD\overline{b}_g$ and $V_1 = A_1a_gB_1b_gC_1\overline{a}_gD_1\overline{b}_g$.

(1) Suppose that N does not involve a_g or b_g. Then

$$k(V,N) = \gamma_V^{-1}\mu_V^{-1}N\mu_{V_1}\gamma_{V_1} = \gamma_V^{-1}(\mu_V^{-1}N\mu_{V_1}N^{-1})N\gamma_{V_1}.$$

Now $\mu_V^{-1}N\mu_{V_1}N^{-1}$ maps $ADCB[a_g,b_g]$ to $\{(A_1D_1C_1B_1)N^{-1}\}[a_g,b_g]$ and fixes each element of X_{g-1}, so that, by Lemma 4, it must involve only a_g and b_g. However, $\mu_V^{-1}N\mu_{V_1}N^{-1}$ fixes a_g and b_g modulo the normal closure of X_{2g-2} in F_{2g}, and so must be the identity. Hence $k(V,N) = \gamma_V^{-1}N\gamma_{V_1} \in M(\Pi_{g-1})$.

This disposes, in particular, of the case $N \in T_1$.

(2) We may now assume that $N = (a,b;b)\theta_{VP} = P\theta_{VP}$. If N involves at most one other letter besides a_g and b_g, then using Lemmas 5 and 6 it follows easily that $k(V,N)$ is nice, and so the result holds by Lemma 7. We now consider a number of cases separately.

Case 2.1. P does not involve a_g or b_g. If $\theta_{VP} = 1$, then this case is covered by (1) above. Otherwise, θ_{VP} must be $a_g \leftrightarrow c$ for some letter

$c \notin \{a_g, b_g, \overline{a}_g, \overline{b}_g\}$. Noting that a_g, b_g are interlocked in VP, we write $VP = A'a_gB'b_gC'\overline{a}_gD'\overline{b}_g$. Let μ be the CP's on a_g, b_g taking VP to $A'B'C'B'[a_g, b_g]$, and let $\gamma \in \mathcal{A}_{2g-2}$ be such that $(A'D'C'B')\gamma = \Pi_{g-1}$. Then

$$k(V,N) = \gamma_V^{-1}\mu_V^{-1}P\theta_{VP}\mu_{V_1}\gamma_{V_1} = (\gamma_V^{-1}\mu_V^{-1}P\mu\gamma)(\gamma^{-1}\mu^{-1}\theta_{VP}\mu_{V_1}\gamma_{V_1}).$$

Repeating the argument given in (1), we see that $\gamma_V^{-1}\mu_V^{-1}P\mu\gamma \in M(\Pi_{g-1})$. Also, θ_{VP} involves only c besides a_g, so that, by (2), $\gamma^{-1}\mu^{-1}\theta_{VP}\mu_{V_1}\gamma_{V_1}$ has a factorisation of the desired form. Hence, the result holds in this case.

We may now assume that P involves exactly one of a_g, b_g. We note, by Corollary 2, that V must contain a subword $a\overline{b}$ or $b\overline{a}$.

Case 2.2. P fixes each element of X_{g-1}. Then P must be one of $(a_g, b; b)$, $(\overline{a}_g, b; b)$, $(b_g, b; b)$ or $(\overline{b}_g, b; b)$, and so a_g, b_g are interlocked in VP.

Suppose that one of the first three possibilities holds. The correcting permutation θ_{VP} in each of these cases is either trivial, or is $a_g \leftrightarrow b^\varepsilon$, $(\varepsilon = \pm 1)$ (for example, if $P = (\overline{a}_g, b; b)$ and ba_g is a subword of V, then $V = A'ba_gBb_gC\overline{a}_gD\overline{b}_g$, where $A = A'b$, and $VP = A'a_gBb_gC\overline{a}_gbD\overline{b}_g$, so that θ_{VP} is $a_g \leftrightarrow \overline{b}$ if $\overline{b} \in B$, and is the identity otherwise). Since only b and a_g are involved in N, the result holds.

Suppose now that $P = (\overline{b}_g, b; b)$. Then we have $V = Aa_gB_1bb_gC\overline{a}_gD\overline{b}_g$ and $VP = Aa_gB_1b_gC\overline{a}_gD\overline{b}_gb$. Since $\theta_V = 1$ we must have $\overline{b} \in A$, so that $V = A_1\overline{b}A_2a_gB_1bb_gC\overline{a}_gD\overline{b}_g$ say, and then $VP = A_1\overline{b}A_2a_gB_1b_gC\overline{a}_gD\overline{b}_gb$. In order to describe θ_{VP}, we must choose the first letter c to the left of $\overline{b}$ in VP so that c, b are interlocked in VP. Thus $\overline{c}$ is in one of A_2, B_1, C or D. The quotient words $V(a_g, b_g, c, b)$ corresponding to these possibilities are $c\overline{b}\overline{c}a_gbb_g\overline{a}_g\overline{b}_g$, $c\overline{b}a_g\overline{c}bb_g\overline{a}_g\overline{b}_g$, $c\overline{b}a_gbb_g\overline{c}\,\overline{a}_g\overline{b}_g$ and $c\overline{b}a_gbb_g\overline{a}_g\overline{c}\overline{b}_g$ respectively. Each of these is equivalent to Π_2, so that $\{a_g, b_g, c, b\}$ is interlocked in V. Thus $k(V,N)$ is nice, and so the required result holds. This disposes of Case 2.2.

The only remaining possibilities are that $b \in \{a_g, b_g, \overline{a}_g, \overline{b}_g\}$, and $a \in L(X_{2g-2})$.

Case 2.3. $b = a_g$ or $b = \overline{a}_g$. Here we note that the effect of P on V is to shift the a_g or $\overline{a}_g$ in V, so that θ_{VP} must be the identity, or of the form $a_g \leftrightarrow c$, for some letter $c \notin \{b_g, \overline{b}_g\}$. If $\theta_{VP} = 1$, or if $c = a^{\pm 1}$, then the result holds, since only a_g and a are involved in N. Otherwise, θ_{VP} is $a_g \leftrightarrow c$ and $c \neq a^{\pm 1}$. Then, for $\varepsilon = \pm 1$.

$$\begin{aligned} P\theta_{VP} = (a, a_g^\varepsilon; a_g^\varepsilon)\theta_{VP} &= \theta_{VP}\{\theta_{VP}^{-1}(a, a_g^\varepsilon; a_g^\varepsilon)\theta_{VP}\} \\ &= \theta_{VP}(a, c^\varepsilon; c^\varepsilon). \end{aligned}$$

Now $k(V,N)^{-1} = k(V_1, N^{-1})$, and $N^{-1} = (a, \overline{c}^{\varepsilon}; \overline{c}^{\varepsilon})\theta_{VP}^{-1}$. Since $(a, \overline{c}^{\varepsilon}; \overline{c}^{\varepsilon})$ does not involve a_g or b_g, the result follows from Case 2.1.

Finally, we have

Case 2.4. $b = b_g$ or $b = \overline{b}_g$. We must consider a number of subcases.

2.4.1. $V = Aa_gBb_gC\overline{a}_gD'a\overline{b}_g$, $P = (a, b_g; b_g)$. Then in V we have $\overline{a} \in A \cup C \cup D'$ (i.e. $\overline{a}$ is a subword of one of A, C, D) since $\theta_V = 1$.

Suppose firstly that $\overline{a} \in D'$. Then $V = Aa_gBb_gC\overline{a}_gD'_2\overline{a}D'_1a\overline{b}_g$ say, so that $VP = Aa_gBb_gC\overline{a}_gD'_1\overline{b}_g\overline{a}D'_2a$. Now θ_{VP} is the product of $b_g \leftrightarrow \overline{a}$ and $a_g \leftrightarrow c$, where c is the first letter to the left of $\overline{a}$ in VP such that $\overline{c} \in D'_2$. Now in V we must have $c \in A \cup C \cup D'_1$ since $\theta_V = 1$. The quotient words $V(a_g, b_g, a, c)$ corresponding to these possibilities are $ca_gb_g\overline{a}_g\overline{a}\,\overline{c}a\overline{b}_g$, $a_gb_gc\overline{a}_g\overline{a}\,\overline{c}a\overline{b}_g$ and $a_gb_g\overline{a}_gc\overline{a}\,\overline{c}a\overline{b}_g$ respectively. Each of these is equivalent to Π_2, and so $k(V,N)$ is nice in this case.

Now suppose that $\overline{a} \in C$. Then $V = Aa_gBb_gC'_1\overline{a}C'_2\overline{a}_gD'a\overline{b}_g$ say, so that $VP = Aa_gBb_gC'_1\overline{b}_g\overline{a}C'_2\overline{a}_gD'a$. Thus θ_{VP} is either $b_g \leftrightarrow \overline{a}$, in which case $k(V,N)$ is nice, or is the product of $b_g \leftrightarrow \overline{a}$ and $a_g \leftrightarrow c$, where $c \in C'_1$ and $\overline{c} \in C'_2 \cup D$. The quotient words $V(a_g, b_g, a, c)$ corresponding to the latter possibility are $a_gb_gc\overline{a}\,\overline{c}\,\overline{a}_ga\overline{b}_g$ and $a_gb_gc\overline{a}\,\overline{a}_g\overline{c}a\overline{b}_g$ and it follows that $k(V,N)$ is nice.

Lastly, suppose that $\overline{a} \in A$. Then $V = A'_1\overline{a}A'_2a_gBb_gC\overline{a}_gD'a\overline{b}_g$ say, so that $VP = A'_1\overline{b}_g\overline{a}A'_2a_gBb_gC\overline{a}_gD'a$. Here θ_{VP} is $\theta_1\theta_2$, where θ_1 is $b_g \leftrightarrow \overline{a}$ and θ_2 is $a_g \leftrightarrow a$, so that N involves only a_g, b_g and a. Consequently $k(V,N)$ is nice. This disposes of case 2.4.1.

2.4.2. $V = Aa_gBb_g\overline{a}C'\overline{a}_gD\overline{b}_g$ and $P = (a, b_g; b_g)$. Since $\theta_V = 1$, we must have $a \in A \cup C' \cup D$.

Suppose firstly that $a \in A$. Then $V = A_1aA_2a_gBb_g\overline{a}C'\overline{a}_gD\overline{b}_g$, say, so that VP is $A_1ab_gA_2a_gB\overline{a}C'a_gD\overline{b}_g$. Then θ_{VP} is $a \leftrightarrow a_g$, and so $k(V,N)$ is nice.

Now suppose that $a \in C'$. Then $V = Aa_gBb_g\overline{a}C'_1aC'_2\overline{a}_gD\overline{b}_g$ say, so that VP is $Aa_gB\overline{a}C'_1ab_gC'_2\overline{a}_gD\overline{b}_g$. Then either $\theta_{VP} = 1$, in which case $k(V,N)$ is nice, or θ_{VP} is $a_g \leftrightarrow c$, where $c \in C'_1$ and $\overline{c} \in C'_2 \cup D$. The quotient words corresponding to the latter possibility are $a_gb_g\overline{a}ca\overline{c}\,\overline{a}_g\overline{b}_g$ and $a_gb_g\overline{a}ca\overline{a}_g\overline{c}\overline{b}_g$ so that $k(V,N)$ is nice.

Lastly, suppose that $a \in D$. Then $V = Aa_gBb_g\overline{a}C'\overline{a}_gD_1aD_2\overline{b}_g$ say, so that VP is $Aa_gB\overline{a}C'a_gD_1ab_gD_2\overline{b}_g$. Then θ_{VP} is $a_g \leftrightarrow c$, where $\overline{c} \in D_2$ and $c \in A \cup C' \cup D_1$. The corresponding quotient words are $ca_gb_g\overline{a}\,\overline{a}_ga\overline{c}\overline{b}_g$, $a_gb_g\overline{a}c\overline{a}_ga\overline{c}\overline{b}_g$ and $a_gb_g\overline{a}\,\overline{a}_gca\overline{c}\overline{b}_g$, so that $k(V,N)$ is nice. This disposes of case 2.4.2.

2.4.3. $V = Aa_gB'ab_gC\overline{a}_gD\overline{b}_g$ and $P = (a, \overline{b}_g; \overline{b}_g)$. Then $a \in A \cup B'$, since $\theta_V = 1$.

Suppose firstly that $\overline{a} \in A$. Then $V = A_1\overline{a}A_2a_gB'ab_gC\overline{a}_gD\overline{b}_g$ say, so that VP is $A_1b_g\overline{a}A_2a_gB'aC\overline{a}_gD\overline{b}_g$. Then θ_{VP} is $a_g \leftrightarrow c$, where $c \in A_1$, $\overline{c} \in A_2 \cup B' \cup C \cup D$. The corresponding quotient words are $c\overline{a}\,\overline{c}a_gab_g\overline{a}_g\overline{b}_g$, $c\overline{a}a_g\overline{c}ab_g\overline{a}_g\overline{b}_g$, $c\overline{a}a_gab_g\overline{c}\,\overline{a}_g\overline{b}_g$ and $c\overline{a}a_gab_g\overline{a}_g\overline{c}\overline{b}_g$, so that $k(V,N)$ is nice.

Now suppose that $\overline{a} \in B'$. Then $V = Aa_gB_1\overline{a}B_2ab_gC\overline{a}_gD\overline{b}_g$ say, so that VP is $Aa_gB_1b_g\overline{a}B_2aC\overline{a}_gD\overline{b}_g$. Then either $\theta_{VP} = 1$, in which case $k(V,N)$ is nice, or θ_{VP} is $a_g \leftrightarrow c$, where $c \in B_1$ and $\overline{c} \in B_2$. The quotient word for the latter possibility is $a_gc\overline{a}\,\overline{c}ab_g\overline{a}_g\overline{b}_g$, so that $k(V,N)$ is nice.

This concludes the proof of the theorem. ■

Let L_g be the complex for the cyclic word Π_g^c (as described in [**9**]). Then L_2 has $4t$ vertices, where $t = 4!2^4$ is the order of the extended symmetric group Ω_4. Thus the quotient complex of L_2 by the obvious Ω_4 action has 4 vertices, representatives of which are the following four vertices of L_2: $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$, $a_1a_2b_1a_1^{-1}b_1^{-1}b_2a_2^{-1}b_2^{-1}$, $a_1b_1a_2a_1^{-1}b_1^{-1}b_2a_2^{-1}b_2^{-1}$ and $a_1b_1a_2b_2a_1^{-1}b_1^{-1}a_2^{-1}b_2^{-1}$. Using the quotient complex, it is straightforward, albeit tedious if done by hand, to compute generators for the stabiliser $M(\Pi_2^c)$ of Π_2^c. This was carried out by the author, and it was verified from this that $M(\Pi_2)$ has generating set $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, where the τ_i satisfy:

$$\begin{aligned}
\tau_1 : a_1 &\to a_1b_1^{-1},\\
\tau_2 : b_1 &\to b_1a_1,\\
\tau_3 : a_1 &\to a_1b_1^{-1}a_2b_2a_2^{-1},\\
a_2 &\to a_2b_2^{-1}a_2^{-1}b_1a_2,\\
b_1 &\to a_2b_2^{-1}a_2^{-1}b_1a_2b_2a_2^{-1},\\
\tau_4 : b_2 &\to b_2a_2,\\
\tau_5 : a_2 &\to a_2b_2^{-1},
\end{aligned}$$

and all generators not explicitly mentioned are left fixed. This generating set was suggested by the corresponding set $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ which is described in [**1**] as a generating set of $M_{g,0}$.

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CANADA

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